Lecture 13 02/28/2018

Maxwell Equations and Electrodynamics (Cont'd)

Some remarks are in order regarding the Coulomb gauge:

(1) Even when $\vec{J}$ is localized, $\vec{J}_t$ that appears in the equation for $\vec{A}$ is not. In general, it is spread out all over the space. Therefore, for a localized $\vec{J}$, it is more useful to use the Lorenz gauge.

(2) The $\vec{B}$ field is always fully transverse since $\vec{\nabla}\cdot\vec{B}=0$. But, in general, the $\vec{E}$ field is not. In the Coulomb gauge, we have $\vec{E}_L=-\vec{\nabla}\Phi$ and $\vec{E}_T=-\frac{\partial\vec{A}}{\partial t}$ since $\vec{\nabla}\cdot\vec{A}=0$.

(3) Since only the causally propagated fields are physical, the actual radiation fields are determined by $\vec{E}_T$ and $\vec{B}$. Both of these are determined by the vector potential $\vec{A}$ in the Coulomb gauge.

(4) In vacuum, when $\rho=0$ and $\vec{J}=0$, we can set $\Phi=0$ in the Coulomb gauge. In this case $\vec{A}$ is the only relevant field that obeys

the homogeneous wave equation $\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t}$.

Green's Function for the Inhomogeneous Wave Equation

In general, the equations for $\vec{A}$ and Φ are inhomogeneous wave equations that have a source term. Using Cartesian coordinates, each of the components of $\vec{A}$ (as well as Φ) satisfies an equation as follows:

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi\, s(\vec{x}, t)$$

In the unbounded space, the general solution to this equation can be written as:

$$\Psi(\vec{x}, t) = \Psi_{hom}(\vec{x}, t) + \int s(\vec{x}', t')\, G(\vec{x}-\vec{x}', t-t')\, d^3x'\, dt'$$

Here, Ψ_{hom} is the general solution to the homogeneous wave equation (i.e., when $s=0$), and G is the Green's function satisfying the following equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x}-\vec{x}', t-t') = -4\pi\, \delta^{(3)}(\vec{x}-\vec{x}')\, \delta(t-t')$$

Defining $\vec{R} \equiv \vec{x}-\vec{x}'$ and $\tau \equiv t-t'$, the Green's function obeys:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{R},\tau) = -4\pi\,\delta^{(3)}(\vec{R})\,\delta(\tau)$$

We note that due to the isotropy of space, G is a function of $R = |\vec{R}|$.

Hence:

$$\nabla^2 G(R,\tau) = \frac{1}{R}\frac{\partial^2}{\partial R^2}\left(R\,G(R,\tau)\right)$$

Let us write the time Fourier transform of G:

$$G(\vec{R},\tau) = \frac{1}{2\pi}\int \underset{\sim}{G}(R,\nu)\,e^{-i\nu\tau}\,d\nu$$

Substituting this in the inhomogeneous wave equation, we find:

$$\left(\nabla^2 + \frac{\nu^2}{c^2}\right)\underset{\sim}{G}(\vec{R},\nu) = -4\pi\,\delta^{(3)}(\vec{R})$$

For $R \neq 0$, the right-hand side of this equation is zero. It then reads:

$$\frac{1}{R}\frac{\partial^2}{\partial R^2}(R\underset{\sim}{G}) + \frac{\nu^2}{c^2}\underset{\sim}{G} = 0 \Rightarrow \frac{\partial^2}{\partial R^2}(R\underset{\sim}{G}) + \frac{\nu^2}{c^2}(R\underset{\sim}{G}) = 0 \Rightarrow$$

$$R\underset{\sim}{G} = A e^{i\frac{\nu}{c}R} + B e^{-i\frac{\nu}{c}R}$$

Thus:

$$\underset{\sim}{G}(R,\nu) = A\frac{e^{\frac{i\nu R}{c}}}{R} + B\frac{e^{-\frac{i\nu R}{c}}}{R}$$

In order to satisfy the equation at $R=0$, we must have $\underline{\underline{B = 1 - A}}$.